

Electromagnetic Excitations of A_n Quantum Hall Droplets

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Abstract

The classical description of A_n internal degrees of freedom is given by making use of the Fock–Bargmann analytical realization. The symplectic deformation of phase space, including the internal degrees of freedom, is discussed. We show that the Moser’s lemma provides a mapping to eliminate the fluctuations of the symplectic structure, which become encoded in the Hamiltonian of the system. We discuss the relation between Moser and Seiberg–Witten maps. One physics applications of this result is the electromagnetic excitation of a large collection of particles, obeying the generalized A_n statistics, living in the complex projective space \mathbf{CP}^k with $U(1)$ background magnetic field. We explicitly calculate the bulk and edge actions. Some particular symplectic deformations are also considered.

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1 Introduction

Quantum Hall effect in higher dimensions has intensively been investigated in the last decade from different point of views [1-10]. This yielded interesting results such as the bosonization [2, 11-13] that has been achieved by making use of the incompressible Hall droplet picture. In this framework, the edge excitations of a quantum Hall droplet are described by a generalized Wess–Zumino–Witten action. Recently, the electromagnetic excitations of a quantum Hall droplet was discussed by Karabali [11] and Nair [12] for the Landau systems in the complex projective space \mathbf{CP}^k . In fact, the corresponding bulk and edge actions were derived [11]. Interestingly, it was shown that the bulk contribution coincides with the $(2k + 1)$ -dimensional Chern–Simons action [12].

The main tool of these developments is the matrix formulation of the lowest Landau level Hall droplet dynamics and the fact that the lowest Landau wavefunctions coincides with the $SU(k + 1)$ coherent states. This provides a simple way to perform the semiclassical analysis in the high magnetic field regime and spinless particles, which form a droplet in the phase space. Along more or less similar lines, the inclusion of spin, color or other degrees of freedom for particles in the phase space formulation has been proposed by Polychronakos [13]. This leads to an elegant gauge generalization of the usual semiclassical droplet picture for scalar particles. The Polychronakos formalism gave rise also the non linear higher dimensional generalization of the gauged Kac-Moody algebra.

Our aim is to analyze the quantum Hall effect in complex projective spaces by incorporating the internal degrees of freedom in the phase space description. We will particularly interested by the generalized A_n statistics introduced firstly by Palev [14], see also [15]. This kind of statistics is associated with the classical Lie algebra A_n and is similar from a mathematical point of view to the generalized spin systems introduced by Randjbar-Daemi *et al.* [16]. Our approach follows the same ideas developed in [13] but is some what different. Indeed, we specify the nature of internal degrees of freedom and more importantly we realize the coupling between the kinematical phase space and the internal degrees of freedom via a symplectic modification procedure. This induces a noncommutativity between two subspaces as well as electromagnetic excitations of the Hall system. We show that the symplectic fluctuations can be eliminated by dressing transformations using the Moser lemma that is a refined version of the celebrated Darboux theorem. In this way, the effect of the symplectic modification becomes encoded in the Hamiltonian of the system.

The paper is organized as follows. In section 2, we recall the main tools needed for the forthcoming analysis. This can be done by defining the generalized statistics associated to the Lie algebras of class A_n and briefly reviewing the quantum Hall effect in \mathbf{CP}^k . In section 3, we give a complete description of the analytical representation of A_n statistics, which uses a representation à la Fock–Bargmann and provides us with phase space description of the internal degrees of freedom for particles obeying A_n statistics. In section 4, the symplectic structure of the phase space of A_n particles living in the manifold \mathbf{CP}^k is introduced. Using a matrix formulation, we derive the action governing the edge excitations of the quantum Hall droplet in the whole phase space. We show that the inclusion of the internal degrees of freedom do not gives rise to any nontrivial dynamics. Thus, in section 5, we purpose a symplectic modification (deformation) of the phase space structure. Using the Moser’s lemma, we give explicitly the dressing transformation, which allows to deal with un-deformed symplectic structure and include the electromagnetic coupling effects in the excitation potential. We also discuss the relation between

the Moser's lemma [17] and Seiberg-Witten map [18], see also [19-20]. In section 6, we give the bulk and edge actions describing the A_n particles in the presence of the electromagnetic interactions. In section 7, we treat the case of particles with A_1 statistics living in two-sphere. We consider the most simplest form of the electromagnetic tensor field (matrix elements constants). In this particular case, a dressing transformation based on the Hilbert-Schmidt orthonormalisation procedure is used to eliminate fluctuations. Obviously, this agrees with one derived from Moser's lemma for small fluctuations. We obtain the action describing the edge excitations and show that, in this special situation, the velocities of the propagation of edge field along the droplet becomes modified due to the electromagnetic coupling. Concluding remarks and comments close this paper.

2 Specifics of A_n statistics and quantum Hall effect in \mathbf{CP}^k

The notion of Bose and Fermi operators has been generalized many years ago to parabosons and parafermions [21]. In this generalization of conventional Bose and Fermi statistics, the paraboson or parafermion algebra is generated by n pairs of creation and annihilation operators (a_{+i}, a_{-i}) , with $i = 1, 2, \dots, n$. They satisfy the trilinear relations which replace the standard bilinear commutation or anti-commutation relations. These are

$$\begin{aligned} [[a_{+i}, a_{-j}]_{\pm}, a_{-k}] &= -2\delta_{ik}a_{-j} \\ [[a_{+i}, a_{+j}]_{\pm}, a_{-k}] &= -2\delta_{ik}a_{+j} \\ [[a_{-i}, a_{-j}]_{\pm}, a_{-k}] &= 0 \end{aligned} \tag{1}$$

where as usual $[x, y]_{\pm} = xy \pm yx$ and the sign \pm refer, respectively, to parabosons and parafermions. The triple relations of parafermion operators give one possible realization of the orthogonal Lie algebra $so(2n+1) = B_n$ [22]. The para-Bose statistics are connected to the orthosymplectic superalgebra $osp(1/2n) = B(0, n)$ [23]. In view of this connection between Lie algebras and parastatistics, a classification of generalized quantum statistics were derived for the classical Lie algebras of class A_n, B_n, C_n and D_n as well as the classical Lie superalgebras [14-15, 24].

The A_n statistics are defined through of n creation and annihilation operators $s_{\pm i}$ satisfying

$$\begin{aligned} [[s_{+i}, s_{-j}], s_{+k}] &= +\delta_{jk}s_{+i} + \delta_{ij}s_{+k} \\ [[s_{+i}, s_{-j}], s_{-k}] &= -\delta_{ik}s_{-j} - \delta_{ij}s_{-k} \end{aligned} \tag{2}$$

implemented by the mutual commutation relations

$$[s_{+i}, s_{+j}] = 0, \quad [s_{-i}, s_{-j}] = 0. \tag{3}$$

It is interesting to stress that the formalism of Lie algebraic statistics is deeply related to the notion of generalized spin systems introduced in [16] for an arbitrary compact group G . In this paper, we consider particles obeying the A_n statistics. The associated algebraic structures will be given in the next section. In particular, we derive the analytical realization of the corresponding representation space, which defines the internal phase space. These particles are assumed to live in the $2k$ -dimensional complex projective space \mathbf{CP}^k .

For \mathbf{CP}^k the magnetic field, which leads to the Landau levels, is proportional to the symplectic two-form, which is

$$\omega_0(z, \bar{z}) = ig_{i\bar{j}} dz^i \wedge d\bar{z}^j \quad (4)$$

where the metric elements are

$$g_{i\bar{j}} = q(1 + \bar{z} \cdot z)^{-2} [(1 + \bar{z} \cdot z)\delta_{ij} - \bar{z}_i z_j] \quad (5)$$

for $i, j = 0, 1, \dots, k$. The lowest Landau levels (LLL) wavefunctions [2]

$$|z_1, z_2, \dots, z_k\rangle = (1 + \bar{z} \cdot z)^{-\frac{q}{2}} \sum_{m_1} \sum_{m_2} \dots \sum_{m_k} \sqrt{\frac{q!}{(q-m)!}} \frac{z_1^{m_1}}{\sqrt{m_1!}} \frac{z_2^{m_2}}{\sqrt{m_2!}} \dots \frac{z_k^{m_k}}{\sqrt{m_k!}} |m_1, m_2, \dots, m_k\rangle. \quad (6)$$

coincide with the \mathbf{CP}^k coherent states, with $m = m_1 + m_2 + \dots + m_k$. Since, the LLL constitute an over complete set with respect to the measure

$$d\mu(\bar{z}, z) = \frac{(q+k)!}{\pi^k q!} \frac{d^2 z_1 d^2 z_2 \dots d^2 z_k}{(1 + \bar{z} \cdot z)^{k+1}}. \quad (7)$$

This provides us with an elegant way to perform the semiclassical analysis, which can be done by associating to any operator acting on the states space

$$\mathcal{F}^{\text{LLL}} = \{|m_1, m_2, \dots, m_k\rangle ; m_1 + m_2 + \dots + m_k \leq q\} \quad (8)$$

a function (or symbol) defined on the phase space coordinate. The commutator between any two operators gives the so-called Moyal star product. It coincides (up to multiplicative factor) with the Poisson bracket in the high magnetic field regime.

3 Generalized A_n spin variables

The semiclassical description of internal degrees of freedom (spin, color, flavor, etc, \dots) presents some conceptual problems because they not possess classical counter parts. However, there exists many ways to realize the spin variables. For instance, the $SU(2)$ spin variables can be viewed as arising from quantization of the coset space $SU(2)/U(1)$ with an appropriate canonical form. This can be generalized to the coset space $\mathbf{CP}^n = SU(n+1)/SU(n) \times U(1)$ and visualized as a realization of the LLL of a particle living in the group manifold $SU(n+1)$ with a magnetic field proportional to the \mathbf{CP}^n symplectic form. Such realization was introduced in [16] where a scheme to generalize the notion of $SU(2)$ spin systems to an arbitrary compact group G was given. This generalization is based on the Holstein–Primakoff representation of spin matrices and used the coherent state method.

In other words, the finite dimensional representation spaces, on which the spin operators (generators of the group G) act, are provided with an over-complete basis labeled by coordinates on a coset space G/H where H includes the Cartan subgroup. According to [13, 16], it follows that the classical phase space, encoding the internal quantum numbers of particles, can be realized by considering the internal quantum numbers arising from the quantization of an internal compact phase space for particles. In this sense, we will show that the generalized A_n statistics algebra introduced by Palev [14], deeply linked to the notion of triple Lie systems introduced by Jacobson [25], provides us with a nice way to define the generalized spin variables and establish a connection between A_n statistics and $SU(n+1)$ spin matrices.

3.1 A_n spin systems

We start by introducing the A_n spin systems. Indeed, they can be defined in terms of the generators $s_{\pm 1}, s_{\pm 2}, \dots, s_{\pm n}$ verifying the triple relations

$$[s_{+i}, s_{-j}, s_{+k}] = +\delta_{jk}s_{+i} + \delta_{ij}s_{+k} \quad (9)$$

$$[s_{+i}, s_{-j}, s_{-k}] = -\delta_{ik}s_{-j} - \delta_{ij}s_{-k} \quad (10)$$

$$[s_{+i}, s_{+j}, s_{\pm k}] = [s_{-i}, s_{-j}, s_{\pm k}] = 0. \quad (11)$$

This description of A_n Lie algebras is a particular case of describing Lie algebras through triple systems initiated by Jacobson [25]. Recall that, a Lie triple system is defined as a subspace of an associative algebra that is closed under the ternary composition $[x, y, z]$. They satisfy the conditions

$$[x, y, z] = -[y, x, z] \quad (12)$$

$$[x, y, z] + [y, z, x] + [z, x, y] = 0 \quad (13)$$

$$[u, v, [x, y, z]] = [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]]. \quad (14)$$

It is easy to see that the ternary composition

$$[x, y, z] = (xyz) - (yxz), \quad (xyz) = xyz + zyx \quad (15)$$

verifies the conditions (12-14). In order to check that the generators $s_{\pm i}$ close a Lie triple algebra under the composition operation (15), it is convenient to consider A_n as subalgebra of the Lie algebra $gl(n+1)$ spanned by the generators e_{ij} , with $e_{ij}e_{kl} = \delta_{ik}e_{jl}$. This can be used to realize $s_{\pm i}$ as

$$s_{+i} = e_{i0} \quad s_{-i} = e_{0i}. \quad (16)$$

Thus, we obtain

$$(s_{+i}s_{-j}s_{+k}) = \delta_{ij}s_{+k} + \delta_{kj}s_{+i}, \quad (s_{-i}s_{+j}s_{-k}) = \delta_{jk}s_{-i} + \delta_{ij}s_{-k} \quad (17)$$

with all other triples are vanishing, now it is clear that (9-10) are verified. Obviously for $[x, y] = xy - yx$, any Lie algebra is a Lie triple system. Indeed, (15) can be rewritten as

$$[x, y, z] = [[x, y], z]$$

which coincides with the definition of generalized A_n statistics (2-3). From a purely algebraic point of view, the Jacobson formulation provides an alternative way to the Chevalley description of classical Lie algebras. From a physical point of view, the Jacobson generators can be seen as creation and annihilation operators satisfying triple relations.

We now consider an Hilbertean representation of the algebra A_n . Let $\mathcal{F}^{\text{spin}}$ be the Hilbert-Fock space on which the generators act. Since the algebra is spanned by n pairs of Jacobson generators, it is natural to assume that the Fock space is given by

$$\mathcal{F}^{\text{spin}} = \{|l_1, l_2, \dots, l_n\rangle, l_i \in \mathbb{N}\} \quad (18)$$

The operators $s_{\pm i}$ are acting on $\mathcal{F}^{\text{spin}}$ [14], see also [26], as

$$s_{-i}|l_1, \dots, l_i, \dots, l_n\rangle = \sqrt{l_i(p+1 - (l_1 + l_2 + \dots + l_n))}|l_1, \dots, l_i - 1, \dots, l_n\rangle \quad (19)$$

$$s_{+i}|l_1, \dots, l_i, \dots, l_n\rangle = \sqrt{(l_i + 1)(p - (l_1 + l_2 + \dots + l_n))}|l_1, \dots, l_i + 1, \dots, l_n\rangle. \quad (20)$$

The dimension of the irreducible representation space is determined by the condition

$$p + 1 - (l_1 + l_2 + \dots + l_n) > 0. \quad (21)$$

$\mathcal{F}^{\text{spin}}$ is generated by a finite number of states satisfying the condition $l_1 + l_2 + \dots + l_n \leq p$, which is a generalized version of the exclusion Pauli principle [14]. This means that no more than p particles can be accommodated in the same quantum state. The integer p defines the order of the statistics. The Fock space dimension is given by

$$\dim \mathcal{F}^{\text{spin}} = \frac{(p + n)!}{p!n!}.$$

Finally, we point out one interesting property of the A_n systems. Indeed, let us introduce the operators

$$b_i^\pm = \frac{s_{\pm i}}{\sqrt{p}}$$

where $i = 1, 2, \dots, n$ and consider p very large. From (19) and (20), one can check that the A_n algebra coincides with n -copies of the Weyl–Heisenberg algebras. In addition, the Jacobson or spin operators reduce to the standard bosonic creation and annihilation operators. Note also that, for $n = 1$ and $p = 1$, one recovers the Fermi creation and annihilation operators.

3.2 Geometry of A_n spin systems

To define the spin variables of generalized A_n systems, we use the coherent state method developed in [16], see also [26–27]. In fact, we will outline the main features needed for our task. The required realization uses a suitably defined Hilbert space of entire analytical functions.

The Jacobson annihilation generators s_{-i} are realized as first order differential operators with respect to a complex variables w_i . This is

$$s_{-i} \longrightarrow \frac{\partial}{\partial w_i}. \quad (22)$$

The key point of such analytical realization lies on the fact that we represent the Fock states $|l_1, l_2, \dots, l_n\rangle$ as power of complex variables w_1, w_2, \dots, w_n , such as

$$|l_1, l_2, \dots, l_n\rangle \longrightarrow C_{l_1, \dots, l_n} w_1^{l_1} w_2^{l_2} \dots w_n^{l_n}. \quad (23)$$

Using the annihilation operator actions on $\mathcal{F}^{\text{spin}}$ and correspondences (22–23), the coefficients C_{l_1, l_2, \dots, l_n} are obtained (up to a normalization constant) as

$$C_{l_1, l_2, \dots, l_n} = \sqrt{\frac{p!}{(p - l)!}} \frac{1}{\sqrt{l_1!} \sqrt{l_2!} \dots \sqrt{l_n!}} \quad (24)$$

where $l = l_1 + l_2 + \dots + l_n$. Using (20) and (24), one can determine the differential action of the Jacobson creation operators, which is

$$s_{+i} \longrightarrow pw_i - w_i \sum_{j=1}^n w_j \frac{d}{dw_j}. \quad (25)$$

Note that, the Jacobson generators act as first order linear differential operators.

On the other hand, an arbitrary vector $|\psi\rangle = \sum_{l_1} \sum_{l_2} \cdots \sum_{l_n} \psi_{l_1, l_2, \dots, l_n} |l_1, l_2, \dots, l_n\rangle$ of $\mathcal{F}^{\text{spin}}$ is realized as

$$\psi(w_1, w_2, \dots, w_n) = \sum_{l_1} \sum_{l_2} \cdots \sum_{l_n} \psi_{l_1, l_2, \dots, l_n} C_{l_1, l_2, \dots, l_n} w_1^{l_1} w_2^{l_2} \cdots w_n^{l_n}. \quad (26)$$

It can be written as the product of the state $|\psi\rangle$ with some ket $\langle \bar{w}| := \langle \bar{w}_1, \bar{w}_2, \dots, \bar{w}_n|$ labeled by the complex conjugate of the variables w_1, w_2, \dots, w_n . This is

$$\psi(w_1, w_2, \dots, w_n) = \mathcal{N} \langle \bar{w}_1, \bar{w}_2, \dots, \bar{w}_n | \psi \rangle \quad (27)$$

where \mathcal{N} is the normalization constant. Taking $|\psi\rangle = |l_1, l_2, \dots, l_n\rangle$, we have

$$\langle \bar{w}_1, \bar{w}_2, \dots, \bar{w}_n | l_1, l_2, \dots, l_n \rangle = \mathcal{N}^{-1} C_{l_1, l_2, \dots, l_n} w_1^{l_1} w_2^{l_2} \cdots w_n^{l_n}. \quad (28)$$

This implies

$$|w_1, w_2, \dots, w_n\rangle = \mathcal{N}^{-1} \sum_{l_1} \sum_{l_2} \cdots \sum_{l_n} \sqrt{\frac{p!}{(p-l)!}} \frac{w_1^{l_1}}{\sqrt{l_1!}} \frac{w_2^{l_2}}{\sqrt{l_2!}} \cdots \frac{w_n^{l_n}}{\sqrt{l_n!}} |l_1, l_2, \dots, l_n\rangle. \quad (29)$$

The normalization constant is given by

$$\mathcal{N} = (1 + |w_1|^2 + |w_2|^2 + \cdots + |w_n|^2)^{\frac{p}{2}} = (1 + \bar{w} \cdot w)^{\frac{p}{2}}. \quad (30)$$

The states (29) are continuous in the labeling, constitute an over complete set with respect to the measure, which can be evaluated as

$$d\mu(\bar{w}, w) = \frac{(p+n)!}{\pi^n p!} \frac{d^2 w_1 d^2 w_2 \cdots d^2 w_n}{(1 + \bar{w} \cdot w)^{n+1}} \quad (31)$$

and then give the $SU(n+1)$ coherent states. This analytical realization enables us to define the spin variables of the A_n phase space systems. It is equipped with a symplectic (Kahler) two-form that makes it into classical phase space. This is realized by introducing the Kahler potential

$$K_0(\bar{w}, w) = \text{Ln}|\langle 0|w\rangle|^{-2} = p \text{Ln}(1 + \bar{w} \cdot w) \quad (32)$$

to define the required closed symplectic two-form as

$$\omega_0(w, \bar{w}) = i g_{i\bar{j}} dw^i \wedge d\bar{w}^j. \quad (33)$$

The corresponding Poisson bracket is

$$\{f, g\} = -i g^{i\bar{j}} \left(\frac{\partial f}{\partial w_i} \frac{\partial g}{\partial \bar{w}_j} - \frac{\partial g}{\partial w_i} \frac{\partial f}{\partial \bar{w}_j} \right). \quad (34)$$

The components of the metric tensor are

$$g_{i\bar{j}} = \frac{\partial^2 K_0(\bar{w}, w)}{\partial w_i \partial \bar{w}_j} = p(1 + \bar{w} \cdot w)^{-2} [(1 + \bar{w} \cdot w) \delta_{ij} - \bar{w}_i w_j]$$

and the matrix elements of its inverse are given by

$$g^{i\bar{j}} = \frac{1}{p} (1 + \bar{w} \cdot w) (\delta_{ij} + w_i \bar{w}_j).$$

As excepted, this is precisely the Kahler structure of the complex projective spaces \mathbf{CP}^n .

4 Hall droplet with internal degrees of freedom

4.1 Phase space description

Most of discussions about the quantum Hall effect ignore the fermionic internal degrees of freedom. Recently, an elegant way to take into account of them was proposed by Polychronakos [13]. According to his ideas, we will consider a dense collection of noninteracting particles living in \mathbf{CP}^k . This defines the kinematical phase space that we enlarge to include the A_n spin variables. The total phase space is then a direct product of the usual phase space and the additional one encoding the internal degrees of freedom of the particles. This is

$$\mathcal{M} = \mathbf{CP}^k \times \mathbf{CP}^n$$

where its dimension is $2(k+n)$. It is endowed with a symplectic two-form of type

$$\omega_0 = ig_{i\bar{j}}(\bar{w}, w)dw^i \wedge d\bar{w}^j + ig_{i\bar{j}}(\bar{z}, z)dz^i \wedge d\bar{z}^j. \quad (35)$$

Clearly, our choice of A_n spin systems living in \mathbf{CP}^k will help us to do our task because the internal and kinematical phase spaces have identical geometrical structures. Moreover, our analysis can also be applied to other systems, for instance, like particles obeying B_n , C_n or D_n statistics living in the Grassmanian manifold $U(k)/U(k_1) \times U(k_2) \times \cdots \times U(k_t)$, with $k_1 + k_2 + \cdots + k_t = k$.

To determine the dynamics of the system under consideration, for a large magnetic field (q large), some tools are needed. These concern the notion of star product, Hushimi (or density) distribution and classical Hamiltonian. We assume also that the order p of the A_n statistics is large. The first needed ingredient is the star product between two functions, which is defined as the mean value of the product of two operators. More precisely, one associate to every operator A acting on the Fock space $\mathcal{F}^{\text{LLL}} \otimes \mathcal{F}^{\text{spin}}$, the function

$$\mathcal{A}(\bar{z}, \bar{w}, z, w) = \langle z, w | A | z, w \rangle. \quad (36)$$

An associative star product of two functions \mathcal{A} and \mathcal{B} is defined by

$$\mathcal{A} \star \mathcal{B} = \langle z, w | AB | z, w \rangle = \int d\mu(\bar{z}', z') d\mu(\bar{w}', w') \langle z, w | A | z', w' \rangle \langle z', w' | B | z, w \rangle \quad (37)$$

where the measures are given by (7) and (31). It follows that the star product between two functions on the phase space can be evaluated to get [27]

$$\mathcal{A} \star \mathcal{B} = \mathcal{A}\mathcal{B} - g^{i\bar{j}} \frac{\partial \mathcal{A}}{\partial w_i} \frac{\partial \mathcal{B}}{\partial \bar{w}_j} - g^{i\bar{j}} \frac{\partial \mathcal{A}}{\partial z_i} \frac{\partial \mathcal{B}}{\partial \bar{z}_j}. \quad (38)$$

Then, the symbol or function associated with the commutator of two operators A and B is given by

$$\langle z, w | [A, B] | z, w \rangle = \{\mathcal{A}, \mathcal{B}\}_\star = -g^{i\bar{j}} \left(\frac{\partial \mathcal{A}}{\partial w_i} \frac{\partial \mathcal{B}}{\partial \bar{w}_j} - \frac{\partial \mathcal{B}}{\partial w_i} \frac{\partial \mathcal{A}}{\partial \bar{w}_j} \right) - g^{i\bar{j}} \left(\frac{\partial \mathcal{A}}{\partial z_i} \frac{\partial \mathcal{B}}{\partial \bar{z}_j} - \frac{\partial \mathcal{B}}{\partial z_i} \frac{\partial \mathcal{A}}{\partial \bar{z}_j} \right) \quad (39)$$

where

$$\{\mathcal{A}, \mathcal{B}\}_\star = \mathcal{A} \star \mathcal{B} - \mathcal{B} \star \mathcal{A} \quad (40)$$

is the so-called the Moyal bracket.

It is well understood that, for large magnetic field, a large collection of particles in the LLL (partially filled) behaves like an incompressible droplet. In the present context, each available state is

also labeled by the A_n spin quantum numbers. Similarly, to the spinless case, we define the density operator as

$$\rho_0 = \rho_0^{\text{LLL}} \otimes \rho_0^{\text{spin}} \quad (41)$$

where ρ_0^{LLL} is the density matrix of M particles in the LLL and the matrix ρ_0^{spin} incorporates the information about the A_n spin states. The symbol associated with the density matrix can be expressed in the coherent state basis to verify that it is a function of $\bar{z} \cdot z$ and $\bar{w} \cdot w$ and with a spherical finite spatial extent. This tells us that we should parameterize the boundary of the droplet by a function $r = r(q \bar{z} \cdot z, p \bar{w} \cdot w)$, such as

$$\rho_0(\bar{z} \cdot z, \bar{w} \cdot w) = \Theta(M - r^2) \quad (42)$$

where Θ is the usual step function. We have a spherical droplet with a radius proportional to \sqrt{M} .

The edge dynamics of spinless droplet is described by the WZW action of type in higher dimensional spaces [2]. According to these previous works, the dynamics of the droplet (42) can be characterized in the following way. One starts with the diagonal density matrix ρ_0 with M states occupied ($M < \dim \mathcal{F}^{\text{LLL}} \times \dim \mathcal{F}^{\text{spin}}$), then the fluctuations preserving the number of states are given by unitary transformation

$$\rho_0 \longrightarrow \rho = U \rho_0 U^\dagger \quad (43)$$

and the equation of motion is the quantum Liouville equation

$$i \frac{\partial \rho}{\partial t} = [V, \rho]. \quad (44)$$

The operator U contains information concerning the edge dynamics and the operator V is the potential generating the excitations on the boundary of the droplet. We consider an oscillator like confining potential as in the spinless case. This is given by

$$\mathcal{V}(\bar{z}, \bar{w}, z, w) = \langle z, w | V | z, w \rangle = q \frac{\bar{z} \cdot z}{1 + \bar{z} \cdot z} + p \frac{\bar{w} \cdot w}{1 + \bar{w} \cdot w}. \quad (45)$$

Here, we have restricted our choice of the quadratic excitation in terms of the internal space variables. This is due to the fact that we choose V so that $[V, \rho_0] = 0$. For large q and p , the excitation potential (45) goes to a superposition of harmonic oscillators. Note that, if we consider an equilibrium configuration, the droplet fills up the phase space to an energy level V_0 . That is the boundary r is such that $\mathcal{V}(r)$ is constant. This defines the droplet boundary as

$$r^2 = q \bar{z} \cdot z + p \bar{w} \cdot w. \quad (46)$$

4.2 Edge effective action

The action that gives the dynamics of edge excitations is

$$S_0 = \int dt \text{Tr} \left\{ \rho_0 U^\dagger (i \partial_t - V) U \right\}. \quad (47)$$

As expected, one can check that the minimization condition for this action leads to the quantum Liouville equation (44). Along the method used by Sakita [28], see also [29], it is more adequate to write the unitary operator U

$$U = e^{+i\Phi}, \quad \Phi^\dagger = \Phi \quad (48)$$

in terms of the abelian field Φ , which will determine the nature of the droplet fluctuations. Reporting (48) in the above action, one can see, after some algebra, that the kinetic part is given by

$$i \int dt \operatorname{Tr} \left(\rho_0 U^\dagger \partial_t U \right) \simeq -\frac{i}{2} \int dt d\mu(\bar{z}, z) d\mu(\bar{w}, w) \{ \Phi, \rho_0 \}_* \partial_t \Phi \quad (49)$$

where we have dropped the terms in $\frac{1}{q^2}$ and $\frac{1}{p^2}$ as well as that related to the total time derivative. In (49), $\Phi = \Phi(\bar{z}, \bar{w}, z, w)$ stands for the symbol associated with the operator Φ . The potential energy can be expanded as

$$\operatorname{Tr} \left(\rho_0 U^\dagger V U \right) = \operatorname{Tr} (\rho_0 V) + i \operatorname{Tr} ([\rho_0, V] \Phi) + \frac{1}{2} \operatorname{Tr} ([\rho_0, \Phi] [V, \Phi]). \quad (50)$$

The first term in r.h.s of (50) is Φ -independent. We drop it since does not contains any information about the dynamics of the edge excitations. The second term vanishes, namely $[\rho_0, V] = 0$. The last term can be evaluated using the correspondence between commutators and Moyal brackets. Thus, the action (47) rewrites as

$$S_0 \simeq -\frac{1}{2} \int dt d\mu(\bar{z}, z) d\mu(\bar{w}, w) \{ \Phi, \rho_0 \}_* [i \partial_t \Phi - \{ \mathcal{V}, \Phi \}_*]. \quad (51)$$

The Moyal brackets involved in the above action can easily be evaluated by making use of (39), (42) and (45). Consequently, we obtain

$$S_0 \approx \frac{1}{2} \int dt \left(\frac{\partial \rho_0}{\partial r^2} \right) \{ (\mathcal{L} \Phi) (\partial_t \Phi) + (\mathcal{L} \Phi)^2 \} \quad (52)$$

where the first order differential operator \mathcal{L} is defined by

$$\mathcal{L} = i(1 - \bar{w} \cdot w)^2 \left(w \cdot \frac{\partial}{\partial w} - \bar{w} \cdot \frac{\partial}{\partial \bar{w}} \right) + i(1 - \bar{z} \cdot z)^2 \left(z \cdot \frac{\partial}{\partial z} - \bar{z} \cdot \frac{\partial}{\partial \bar{z}} \right). \quad (53)$$

Since the derivative of the density function is a delta function, the action is defined on the boundary of the droplet and involved only the time derivative of Φ and the tangential derivative $\mathcal{L} \Phi$. It is similar to that derived in the spinless case for \mathbf{CP}^k [2], the Bergman ball \mathbf{B}^k and flag manifold \mathbf{F}_2 [10]. More interestingly, the equation of motion arising from (52) for the edge field Φ is

$$\mathcal{L}(\partial_t \Phi + \mathcal{L} \Phi) = 0. \quad (54)$$

This shows that any nontrivial dynamics arose out in this description. The kinematical phase space and the internal spin degrees of freedom dynamics are disconnected. In this respect, it is interesting to couple spin and kinematical degrees of freedom. This coupling can be realized by introducing a nonzero phase space structure between two relevant spaces. More precisely, this can be achieved by a symplectic deformation of two-form (35).

5 Noncommutative phase space and Seiberg–Witten map

5.1 Deformed symplectic structure

According to the total phase space realization given in (35), one can write the corresponding symplectic two-form as

$$\omega_0 = \sum_{i=1}^r d\xi^i \wedge d\xi^{i+r} = \sum_{i=1}^r dq^i \wedge dp^i \quad (55)$$

where $r = k + n$ and the real canonical coordinates ξ^i ($i = 1, 2, \dots, 2r$) are defined by

$$\begin{aligned}\frac{w^i}{\sqrt{1 + \bar{w} \cdot w}} &= \frac{1}{\sqrt{2p}}(\xi^i + i\xi^{i+r}), & i = 1, 2, \dots, n \\ \frac{z^i}{\sqrt{1 + \bar{z} \cdot z}} &= \frac{1}{\sqrt{2q}}(\xi^{i+n} + i\xi^{i+r+n}), & i = 1, 2, \dots, k.\end{aligned}$$

Assuming that (55) is modified due to the presence of some external electromagnetic background. More precisely, we replace the canonical two-form ω_0 by a closed new one, such as

$$\omega = \omega_0 - \frac{1}{2} \sum_{i,j=1}^r \mathcal{B}_{ij} d\xi^i \wedge d\xi^j + \frac{1}{2} \sum_{i,j=1}^r \mathcal{E}_{ij} d\xi^{i+r} \wedge d\xi^{j+r} \quad (56)$$

where the antisymmetric tensors \mathcal{B}_{ij} and \mathcal{E}_{ij} are phase space variables dependants. In addition, we also assume that the tensor \mathcal{B}_{ij} and \mathcal{E}_{ij} , encoding the deformation, are functions of the phase space coordinates ξ^i and ξ^{i+r} ($i = 1, 2, \dots, r$), respectively. These are

$$\begin{aligned}\mathcal{B}_{ij} &\equiv \mathcal{B}_{ij}(\xi^1, \xi^2, \dots, \xi^r) \\ \mathcal{E}_{ij} &\equiv \mathcal{E}_{ij}(\xi^{r+1}, \xi^{r+2}, \dots, \xi^{2r}).\end{aligned}$$

The $U(1)$ connections \mathcal{B} and \mathcal{E}

$$\mathcal{B} = dA \quad \mathcal{E} = d\bar{A} \quad (57)$$

are given in terms of the gauge fields

$$A = \sum_{i=1}^r A_i(q) dq^i, \quad \bar{A} = \sum_{i=1}^r \bar{A}_i(p) dp^i \quad (58)$$

where the notation bar does not mean the complex conjugate. Alternatively, in compact form, (56) can be written as

$$\omega = \frac{1}{2} \omega_{IJ} d\xi^I \wedge d\xi^J \quad (59)$$

or equivalently, in matrix representation as

$$(\omega) = \begin{pmatrix} -\mathcal{B} & \mathbf{1}_{r \times r} \\ -\mathbf{1}_{r \times r} & \mathcal{E} \end{pmatrix}$$

where $\mathbf{1}_{r \times r}$ stands for the identity matrix. Two-form ω is nondegenerate, i.e. $\det \omega \neq 0$, when the antisymmetric tensors \mathcal{E}_{ij} and \mathcal{B}_{ij} satisfy the condition $\det(\mathbf{1}_{r \times r} - \mathcal{E}\mathcal{B}) \neq 0$. We assume that such condition is satisfied. To find the classical equations of motion and establish the connection between the classical and quantum theory, it is necessary to define the Poisson brackets associated with the new phase space geometry in a consistent way. Indeed, recalling that the Poisson brackets for the coordinates on the phase space are the inverse of the symplectic form as matrix. This gives

$$\{\mathcal{F}, \mathcal{G}\} = \sum_{I,J} (\omega^{-1})^{IJ} \frac{\partial \mathcal{F}}{\partial \xi^I} \frac{\partial \mathcal{G}}{\partial \xi^J} \quad (60)$$

where $(\omega^{-1})^{IJ}$ is the inverse matrix of ω_{IJ} and $(\mathcal{F}, \mathcal{G})$ are two functions defined on the phase space. Using the matricial form of ω , it is easy to determine the corresponding inverse. One can see that the Poisson brackets take the simple form

$$\{\mathcal{F}, \mathcal{G}\} = \sum_{ik} (\omega_1^{-1})_{ik} \frac{\partial \mathcal{F}}{\partial \xi^i} \left[\frac{\partial \mathcal{G}}{\partial \xi^{k+r}} - \sum_j \mathcal{E}_{kj} \frac{\partial \mathcal{G}}{\partial \xi^j} \right] - (\omega_2^{-1})_{ik} \frac{\partial \mathcal{F}}{\partial \xi^{i+r}} \left[\frac{\partial \mathcal{G}}{\partial \xi^k} - \sum_j \mathcal{B}_{kj} \frac{\partial \mathcal{G}}{\partial \xi^{j+r}} \right] \quad (61)$$

where the matrix elements of the matrices are defined by

$$\begin{aligned} (\omega_1)_{ij} &= \delta_{ij} - \sum_k \mathcal{E}_{ik} \mathcal{B}_{kj} \\ (\omega_2)_{ij} &= \delta_{ij} - \sum_k \mathcal{B}_{ik} \mathcal{E}_{kj}. \end{aligned} \quad (62)$$

These can be read in matrices form as $\omega_1 = \mathbf{1}_{r \times r} - \mathcal{E} \mathcal{B}$ and $\omega_2 = \mathbf{1}_{r \times r} - \mathcal{B} \mathcal{E}$, respectively. It follows that the modified canonical Poisson brackets are

$$\begin{aligned} \{\xi^i, \xi^j\} &= - \sum_k (\omega_1^{-1})_{ik} \mathcal{E}_{kj} \\ \{\xi^{i+r}, \xi^{j+r}\} &= \sum_k (\omega_2^{-1})_{ik} \mathcal{B}_{kj} \\ \{\xi^i, \xi^{j+r}\} &= (\omega_1^{-1})_{ij} = (\omega_2^{-1})_{ji}. \end{aligned} \quad (63)$$

According to the modification of the symplectic structure of the phase space, we introduce the vector fields $X_{\mathcal{F}}$ associated to a given functional $\mathcal{F}(\xi^i, \xi^{i+r})$

$$X_{\mathcal{F}} = \sum_i X^i \frac{\partial}{\partial \xi^i} + Y^i \frac{\partial}{\partial \xi^{i+r}} \quad (64)$$

such that the interior contraction of ω with $X_{\mathcal{F}}$ gives

$$i(X_{\mathcal{F}})\omega = d\mathcal{F}. \quad (65)$$

A straightforward calculation leads

$$\begin{aligned} X^i &= \sum_j (\omega_1^{-1})_{ij} \left(\frac{\partial \mathcal{F}}{\partial \xi^{j+r}} - \sum_k \mathcal{E}_{jk} \frac{\partial \mathcal{F}}{\partial \xi^k} \right) \\ Y^i &= - \sum_j (\omega_2^{-1})_{ij} \left(\frac{\partial \mathcal{F}}{\partial \xi^j} - \sum_k \mathcal{B}_{jk} \frac{\partial \mathcal{F}}{\partial \xi^{k+r}} \right) \end{aligned} \quad (66)$$

and one can check

$$i(X_{\mathcal{F}})i(X_{\mathcal{G}})\omega = \{\mathcal{F}, \mathcal{G}\}. \quad (67)$$

As illustration of the above construction, let us consider the configuration with $r = 2$, \mathcal{E}_{ij} and \mathcal{B}_{ij} are two constants, such as

$$\mathcal{E}_{ij} = \theta \epsilon_{ij}, \quad \mathcal{B}_{kj} = \bar{\theta} \epsilon_{ij} \quad (68)$$

where ϵ_{ij} is the usual antisymmetric tensor ($\epsilon_{12} = -\epsilon_{21} = 1$). In this case, (63) becomes

$$\begin{aligned}\{\xi^i, \xi^j\} &= -\frac{\theta}{1 + \theta\bar{\theta}}\epsilon_{ij} \\ \{\xi^{i+r}, \xi^{j+r}\} &= \frac{\bar{\theta}}{1 + \theta\bar{\theta}}\epsilon_{ij} \\ \{\xi^i, \xi^{j+r}\} &= \frac{1}{1 + \theta\bar{\theta}}\delta_{ij}.\end{aligned}\tag{69}$$

which are reflecting a deviation from the canonical brackets. At this stage, it is remarkable that the modified symplectic form (56) and corresponding Poisson brackets can be converted in the canonical forms. This can be achieved by a dressing transformation based on the Moser's lemma [17].

5.2 Symplectic dressing and Moser's lemma

As mentioned above, the Moser's lemma provides us with a nice procedure to locally eliminate the fluctuations of the symplectic structure. Let us first revisit the derivation of such lemma in order to give as possible as general expression of the dressing transformation, which maps the modified two-form to the original one. Indeed, we consider the most general case where the matrix elements

$$(\omega_0)_{IJ} \equiv (\omega_0)_{IJ}(\xi)$$

are nonconstants. We assume that the fluctuation is induced by a closed two-form

$$F = dA, \quad F = A_I(\xi)d\xi^I$$

where the summation over repeated indices is understood throughout this subsection.

The Moser's lemma tells us that always there exists a diffeomorphism on the phase space ϕ whose pullback maps ω to ω_0

$$\phi^*(\omega_0 + F) = \omega_0\tag{70}$$

which is

$$\phi : \xi^I \mapsto \phi(\xi^I), \quad \frac{\partial \phi(\xi^K)}{\partial \xi^I} \frac{\partial \phi(\xi^L)}{\partial \xi^J} \omega_{KL}(\phi(\xi)) = \omega_{IJ}(\xi).\tag{71}$$

To find out this change of coordinates, one can start by defining a family of one parameter of symplectic forms

$$\omega(t) = \omega_0 + tF\tag{72}$$

interpolating ω_0 and $\omega_0 + F$ for $t = 0$ and $t = 1$, respectively, with $0 \leq t \leq 1$. Note that, t is just an affine parameter labeling the flow generated by a smooth t -dependent vector field $X(t)$. Accordingly, one also define a family of diffeomorphisms

$$\phi^*(t)\omega(t) = \omega_0\tag{73}$$

satisfying $\phi^*(t = 0) = \text{id}$ and the diffeomorphism $\phi^*(t = 1)$ will be then the required solution of our problem, i.e. (70). Differentiating (73), one check that $X(t)$ must satisfy the identity

$$0 = \frac{d}{dt} [\phi^*(t)\omega(t)] = \phi^*(t) \left[L_{X(t)}\omega(t) + \frac{d\omega(t)}{dt} \right]\tag{74}$$

where $L_{X(t)}$ denotes the Lie derivative of $X(t)$. Using the Cartan identity $L_X = \iota_X \circ d + d \circ \iota_X$ and the fact that $d\omega(t) = 0$ to obtain

$$\phi^*(t) \{d[\iota_{X(t)}\omega(t)] + F\} = 0 \quad (75)$$

where ι_X stands for interior contraction as above. It follows that $X(t)$ is satisfying

$$\iota_{X(t)}\omega(t) + A = 0. \quad (76)$$

Therefore, the components of $X(t)$ are given by

$$X^I(t) = -A_J \omega^{-1 JI}(t). \quad (77)$$

For small fluctuations of the symplectic structure, i.e. $F \ll \omega_0$, one can write the inverse of ω as

$$\omega^{-1}(t) = \omega_0^{-1} - t\omega_0^{-1}F\omega_0^{-1} + t^2\omega_0^{-1}F\omega_0^{-1}F\omega_0^{-1} + \dots \quad (78)$$

This determines the components of $X(t)$ in terms of the $U(1)$ connection A and its derivatives. It allows us to write down the explicit form of the transformation ϕ . Indeed, since the t -evolution of $\omega(t)$ is governed by the first order differential equation

$$[\partial_t + X(t)]\omega(t) = 0 \quad (79)$$

it is easy to see

$$[\exp(\partial_t + X(t)) \exp(-\partial_t)]\omega(t+1) = \omega(t). \quad (80)$$

Using this, one can verify the relation

$$[\exp(\partial_t + X(t)) \exp(-\partial_t)]|_{(t=0)}(\omega_0 + F) = \phi^*(\omega_0 + F) = \omega_0 \quad (81)$$

from which one obtains the pull-back mapping ω to ω_0 . This is

$$\phi^* = \text{id} + X(0) + \frac{1}{2}(\partial_t X)(0) + \frac{1}{2}X^2(0) + \dots \quad (82)$$

More explicitly, using (77) and (78), the contribution of the second term in (82) reads as

$$X(0) = \omega_0^{-1 IJ} A_J \partial_I. \quad (83)$$

The contribution of the third term in (82) is given by

$$\frac{1}{2}(\partial_t X)(0) = -\frac{1}{2}(\omega_0^{-1} F \omega_0^{-1})^{IJ} A_J \partial_I. \quad (84)$$

Similarly, the last term in (82) leads

$$\frac{1}{2}X^2(0) = \frac{1}{2}(\omega_0^{-1 IJ} A_J \partial_I)(\omega_0^{-1 I'J'} A_{J'} \partial_{I'}). \quad (85)$$

Finally, in terms of local coordinates, the coordinate transformation ϕ whose pullback maps $\omega_0 + F \longrightarrow \omega_0$ takes the form

$$\phi(\xi^L) = \xi^L + \xi_1^L + \xi_2^L + \dots \quad (86)$$

where the second term reads as

$$\xi_1^L = \omega_0^{-1LJ} A_J \quad (87)$$

and the third one is

$$\xi_2^L = -\frac{1}{2}\omega_0^{-1LK} F_{KL'} \omega_0^{-1L'J} A_J + \frac{1}{2}\omega_0^{-1IJ} A_J (\partial_I \omega_0^{-1LJ'}) A_{J'} + \frac{1}{2}\omega_0^{-1IJ} A_J \omega_0^{-1LJ'} (\partial_I A_{J'}). \quad (88)$$

Using the relations

$$\partial_{J'} A_{I'} = (\partial_{J'} \omega_{0I'I}) \xi_1^I + \omega_{0I'I} (\partial_{J'} \xi_1^I) \quad (89)$$

$$\partial_I \omega_0^{-1LJ'} = -\omega_0^{-1LJ''} (\partial_I \omega_{0J''K}) \omega_0^{-1KJ'} \quad (90)$$

and the antisymmetry property of the symplectic form, recall that ω_0 is assumed closed and with non constants matrix elements, one can verify

$$\begin{aligned} \xi_2^L &= -\omega_0^{-1LK} F_{KL'} \xi_1^{L'} + \frac{1}{2}\omega_0^{-1LK} \omega_0^{-1MJ} A_J \omega_0^{-1NJ'} A_{J'} \partial_M \omega_{0NK} + \\ &+ \frac{1}{2}\omega_0^{-1LK} \omega_0^{-1MS} A_S \omega_{0MN} \partial_K (\omega_0^{-1NS'} A_{S'}). \end{aligned} \quad (91)$$

It is remarkable that the above dressing transformation coincides with the Susskind map derived in connection with the quantum Hall systems and noncommutative Chern–Simons theory [30]. Moreover, it leads to the very familiar Seiberg–Witten map [18] used in the context of the string and noncommutative gauge theories.

5.3 Seiberg–Witten and/or Susskind map

One can see from (86-88) that the dressing transformation can be written as

$$\phi(\xi^L) = \xi^L + \hat{A}^L \quad (92)$$

where the coordinates fluctuations is the sum of the quantities (87) and (91) given by

$$\begin{aligned} \hat{A}^L &= \omega_0^{-1LK} \left[A_K - F_{KL'} \omega_0^{-1L'M} A_M + \frac{1}{2}\omega_0^{-1MJ} A_J \omega_0^{-1NJ'} A_{J'} \partial_M \omega_{0NK} \right] \\ &+ \frac{1}{2}\omega_0^{-1LK} \left[\omega_0^{-1MS} A_S \omega_{0MN} \partial_K (\omega_0^{-1NS'} A_{S'}) \right]. \end{aligned} \quad (93)$$

The relation (92) can be viewed as a generalization of Susskind map [30]. It encodes the geometrical fluctuations induced by the external magnetic field F .

Moreover, (92) coincides also with the Seiberg–Witten map in a curved manifold for a noncommutative abelian gauge theory [19]. Indeed, under the gauge transformation

$$A \longrightarrow A + d\Lambda \quad (94)$$

the components (93) transform as

$$\hat{A}^L \longrightarrow \hat{A}^L + \omega_0^{-1LJ} \partial_J \hat{\Lambda} + \{\hat{A}^L, \hat{\Lambda}\} + \dots \quad (95)$$

where the noncommutative gauge parameter $\hat{\Lambda}$

$$\hat{\Lambda} = \Lambda + \frac{1}{2}\omega_0^{-1IJ} A_J \partial_I \Lambda + \dots \quad (96)$$

is expressed in terms of Λ and the abelian connection A . (94), (95) and (96) are the semiclassical version of the Seiberg–Witten map. The connection \hat{A} is an induced noncommutative gauge field given in terms of its commutative counter part A . This establish clearly the above mentioned relation between Moser’s lemma and Seiberg–Witten map and gives a correspondence between the symplectic deformations and noncommutative gauge theories.

At this stage, we have the necessary tools to map two-form (56) to (55). Indeed, using (86) together with (87) and (91), one obtains the coordinates change such that $\omega_0 + \mathcal{E} + \mathcal{B}$ takes the canonical form

$$\omega_0 + F = dQ^i \wedge dP^i \quad (97)$$

where the new phase space variables Q^i and P^i are given by

$$Q^i = \phi^{-1}(q^i) = q^i + \bar{A}_i(p) - \sum_j A_j(q) \left[\mathcal{E}_{ij}(p) - \frac{1}{2} \frac{\partial \bar{A}_j(p)}{\partial p_i} \right] + \dots \quad (98)$$

$$P^i = \phi^{-1}(p^i) = p^i - A_i(q) + \sum_j \bar{A}_j(p) \left[\mathcal{B}_{ij}(q) + \frac{1}{2} \frac{\partial A_j(q)}{\partial q_i} \right] + \dots \quad (99)$$

It is interesting to note that for $\bar{A}_i(p) = 0$, respectively $A_i(q) = 0$, we recover the Darboux transformations. On the other hand, for $r = 2$ when the gauge potentials (57) are defined as

$$A + \bar{A} = -\frac{1}{2}(\bar{\theta}\epsilon_{ij}q_i dq_j - \theta\epsilon_{ij}p_i dp_j) \quad (100)$$

which is corresponding to constant electromagnetic fields F given by (68), the dressing transformation (98-99) gives

$$Q^i = (1 + \frac{3}{8}\theta\bar{\theta})q^i + \frac{\theta}{2} \sum_k \epsilon_{ki} p^k \quad (101)$$

$$P^i = (1 + \frac{3}{8}\theta\bar{\theta})p^i + \frac{\bar{\theta}}{2} \sum_k \epsilon_{ki} q^k. \quad (102)$$

This result will be compared with that will be derived in section 7, see equation (122), when we will deal with an exact dressing transformation (Hilbert–Schmidt orthonormalization procedure mentioned above), which is applicable in some few particular cases.

6 Electromagnetic excitations of A_n quantum Hall droplet

6.1 Hamiltonian and induced electromagnetic interaction

In modifying the symplectic structure, the dynamics becomes described by two-form $\omega_0 + F$. The dressing transformation converts the dynamical system of $(\omega_0 + F, \mathcal{V}) \Big|_{qp}$ to $(\omega_0, \mathcal{V}_A) \Big|_{QP}$ where we use the old symplectic form but a different Hamiltonian. This latter, can be obtained by simply replacing

the old phase space variables in terms of the new ones. In this respect, inverting (98) and (99), one obtains

$$q^i = \phi(Q^i) = Q^i - \bar{A}_i(P) + \sum_j A_j(Q) \left[\mathcal{E}_{ij}(P) - \frac{1}{2} \frac{\partial \bar{A}_j(P)}{\partial P_i} \right] + \dots \quad (103)$$

$$p^i = \phi(P^i) = P^i + A_i(Q) - \sum_j \bar{A}_j(P) \left[\mathcal{B}_{ij}(Q) + \frac{1}{2} \frac{\partial A_j(Q)}{\partial Q_i} \right] + \dots \quad (104)$$

Using this, to obtain the required Hamiltonian form to the second order in the A 's, which is

$$\begin{aligned} \mathcal{V}_A = & \mathcal{V}(Q, P) - \sum_i (\bar{A}_i \bar{u}_i - A_i u_i) + \frac{1}{2} \sum_{ij} \left[\bar{A}_i \bar{A}_j \frac{\partial \bar{u}_i}{\partial Q_j} + A_i A_j \frac{\partial u_i}{\partial P_j} - 2 \bar{A}_i A_j \frac{\partial u_j}{\partial Q_i} \right] \\ & + \sum_{ij} A_j \left[\mathcal{E}_{ij} - \frac{1}{2} \frac{\partial \bar{A}_j}{\partial P_i} \right] \bar{u}_i - \sum_{ij} \bar{A}_j \left[\mathcal{B}_{ij} + \frac{1}{2} \frac{\partial A_j}{\partial Q_i} \right] u_i + \dots \end{aligned} \quad (105)$$

where

$$u_i = \frac{\partial \mathcal{V}}{\partial P_i}, \quad \bar{u}_i = \frac{\partial \mathcal{V}}{\partial Q_i}. \quad (106)$$

Here again bar does not stand for complex conjugate. It is clear that, the dressing transformation eliminates the fluctuations of the symplectic form, which become encoded in the Hamiltonian. In (105), the excitation potential \mathcal{V} is

$$\mathcal{V} \equiv \mathcal{V}(Q, P) = \frac{1}{2} \sum_{i=1}^r [Q_i^2 + P_i^2] \quad (107)$$

and the quantities defined by (106) reduces to $u_i = P_i$ and $\bar{u}_i = Q_i$.

6.2 Generalized WZW action

The analysis developed in the previous section can be applied to derive the action describing the electromagnetic interaction of a higher dimensional Hall system. Following similar lines as before, the action is given by

$$S = \int dt \text{Tr} \left(i \rho_0 U^\dagger \partial_t U - \rho_0 U^\dagger V_A U \right) \quad (108)$$

where V_A is the operator associated to the Hamiltonian function given by (107). Note that, the kinetic part of the action remains unchanged. Using similar tools as above, the action can be rewritten as

$$S = S_0 + S_A \quad (109)$$

where the unperturbed action S_0 is given by (52) and the electromagnetic coupling contribution takes the form

$$S_A = - \int dt \left[\rho_0 \star \mathcal{V}_A - (\mathcal{L}\Phi) \frac{\partial \rho_0}{\partial r^2} \mathcal{V}_A \right]. \quad (110)$$

The above action can also be written as the sum of two terms: edge and bulk contributions. Indeed, the Hamiltonian \mathcal{V}_A can be expressed as

$$\mathcal{V}_A = \mathcal{V}_A^b + \mathcal{V}_A^e \quad (111)$$

where the first term is

$$\begin{aligned} \mathcal{V}_A^b = & \mathcal{V} - \sum_i (\bar{A}_i \bar{u}_i - A_i u_i) + \sum_{ij} \left[\bar{A}_i u_j \frac{\partial A_j}{\partial Q_i} \right] \\ & + \sum_{ij} A_j \left[\mathcal{E}_{ij} - \frac{1}{2} \frac{\partial \bar{A}_j}{\partial P_i} \right] \bar{u}_i - \sum_{ij} \bar{A}_j \left[\mathcal{B}_{ij} + \frac{1}{2} \frac{\partial A_j}{\partial Q_i} \right] u_i + \dots \end{aligned} \quad (112)$$

and the second reads as

$$\mathcal{V}_A^e = \frac{1}{2} \sum_{ij} \left[\frac{\partial(\bar{A}_i \bar{A}_j \bar{u}_i)}{\partial Q_i} + \frac{\partial(A_i A_j u_i)}{\partial P_j} - 2 \frac{\partial(\bar{A}_i A_j u_j)}{\partial Q_i} \right]. \quad (113)$$

It follows that the action (110) can be separated into two parts

$$S_A = S_A^{\text{bulk}} + S_A^{\text{edge}} \quad (114)$$

where the edge contribution is given by

$$S_A^{\text{edge}} = - \int dt \left[\rho_0 \star \mathcal{V}_A^e - (\mathcal{L}\Phi) \frac{\partial \rho_0}{\partial r^2} \mathcal{V}_A \right] \quad (115)$$

and the bulk term is

$$S_A^{\text{bulk}} = - \int dt \rho_0 \star \mathcal{V}_A^b. \quad (116)$$

Note that, the quantity (113) involves only derivatives. Then, by expanding the star product in the first term in (115) and integrating, one can see that it produces a boundary contribution. The second term in (115) is obviously a boundary contribution because it involves the derivative of the density, i.e. the derivative of the density gives a delta function defined on the edge. Clearly, in the absence of electromagnetic field $A = 0$, the action (109) reduces to S_0 as it should be.

To close this section two remarks are in order. First, the electromagnetic excitation of \mathbf{CP}^k quantum Hall droplet, with A_n spin degrees of freedom, is generated here from a purely symplectic point of view. This gives an alternative description to the results presented in [11] where a similar issue was investigated. Secondly, our model can easily be adapted to some geometries and situations with other kinds of internal degrees of freedom.

7 A_1 quantum Hall droplet in sphere

7.1 Excitation potential

In this section, we consider the most simplest case where the particles are living in the sphere \mathbf{S}^2 and the internal degrees of freedom are described by A_1 spin algebra. Here the total phase space \mathcal{M} is four-dimensional space. As far as the electromagnetic excitation (or symplectic modification) is concerned, we will restrict ourself to two-form given by (68). Instead of the dressing transformation based on the Moser's lemma, we make use of another method, which gives in this particular case an exact coordinates change. This is based on the so-called Hilbert–Schmidt orthonormalization procedure. In the present case, we rewrite the Poisson brackets (69) as

$$\{q^i, q^j\} = - \frac{\theta}{1 + \theta \bar{\theta}} \epsilon_{ij}$$

$$\begin{aligned}\{p^i, p^j\} &= \frac{\bar{\theta}}{1 + \theta\bar{\theta}} \epsilon_{ij} \\ \{q^i, p^j\} &= \frac{1}{1 + \theta\bar{\theta}} \delta_{ij}\end{aligned}\tag{117}$$

where $i, j = 1, 2$, they reflect a deviation from the canonical brackets. In addition, they traduce the noncommutativity between the coordinates and momenta of the internal and kinematical phase spaces.

The excitation potential (45) coincides for $r = 2$ with a bidimensional harmonic oscillator. It is simply verified that the deformed symplectic two-form $\omega_0 + \mathcal{B} + \mathcal{E}$ can be converted in a canonical one by using an analog of the Hilbert–Schmidt transformation

$$\begin{aligned}Q^i &= a q^i + \frac{1}{2} b \theta \sum_k \epsilon_{ki} p^k \\ P^i &= c p^i + \frac{1}{2} d \bar{\theta} \sum_k \epsilon_{ki} q^k\end{aligned}\tag{118}$$

where

$$a = c = \frac{1}{b} = \frac{1}{d} = \frac{1}{\sqrt{2}} \sqrt{1 + \sqrt{1 + \theta\bar{\theta}}}.\tag{119}$$

We assume that $1 + \theta\bar{\theta} > 0$. Consequently, the Poisson brackets (117) give the canonical ones

$$\begin{aligned}\{Q^i, Q^j\} &= 0 \\ \{P^i, P^j\} &= 0 \\ \{Q^i, P^j\} &= \delta_{ij}\end{aligned}\tag{120}$$

and the symplectic two-form becomes

$$\omega = \sum_i dQ^i \wedge dP^i.\tag{121}$$

It is interesting to notice that, for small values of θ and $\bar{\theta}$, (118) and (119) give

$$\begin{aligned}Q^i &= (1 + \frac{1}{8} \theta \bar{\theta}) q^i + \frac{\theta}{2} \sum_k \epsilon_{ki} p^k \\ P^i &= (1 + \frac{1}{8} \theta \bar{\theta}) p^i + \frac{\bar{\theta}}{2} \sum_k \epsilon_{ki} q^k\end{aligned}\tag{122}$$

which are sensitively similar to (101) and (102) those obtained by using Moser’s lemma. Inverting the transformation (118), we have

$$\begin{aligned}q^i &= \frac{a}{\sqrt{1 + \theta\bar{\theta}}} \left[Q^i + \frac{\theta}{2a^2} \sum_k \epsilon_{ik} P^k \right] \\ p^i &= \frac{a}{\sqrt{1 + \theta\bar{\theta}}} \left[P^i + \frac{\bar{\theta}}{2a^2} \sum_k \epsilon_{ik} Q^k \right].\end{aligned}\tag{123}$$

The excitation potential (45) for $r = 2$ becomes

$$\mathcal{V} = \frac{a^2}{2(1 + \theta\bar{\theta})} \left[\sum_i (1 + \frac{\theta^2}{4a^4}) P^i P^i + (1 + \frac{\bar{\theta}^2}{4a^4}) Q^i Q^i + (\frac{\theta}{a^2} - \frac{\bar{\theta}}{a^2}) \sum_j \epsilon_{ij} Q^i P^j \right].\tag{124}$$

Evidently the transformation (118) also eliminates the modification of the symplectic structure, which become incorporated in the excitation potential. For our purpose, the expression (124) can be cast in a more appropriate form by introducing the variables

$$z_{\pm} = \frac{\sqrt{\Delta}}{2} \left[(Q^1 \pm iQ^2) + \frac{i}{\Delta} (P^1 \pm iP^2) \right] \quad (125)$$

where

$$\Delta = \sqrt{\frac{4a^4 + \bar{\theta}^2}{4a^4 + \theta^2}}. \quad (126)$$

Indeed, substituting (125) in (124), we obtain

$$\mathcal{V} = \omega_+ z_+ \bar{z}_+ + \omega_- z_- \bar{z}_- \quad (127)$$

where

$$\omega_{\pm} = \frac{\sqrt{(4a^4 + \theta^2)(4a^4 + \bar{\theta}^2)}}{4a^2(1 + \theta\bar{\theta})} \mp \frac{\theta - \bar{\theta}}{2(1 + \theta\bar{\theta})}. \quad (128)$$

Note that, two-form (121) rewrites as

$$\omega = i(dz_+ \wedge d\bar{z}_+ + dz_- \wedge d\bar{z}_-). \quad (129)$$

Upon quantization, all canonical variables become Heisenberg operators satisfying commutation rules according to the canonical prescription: (Poisson bracket \longrightarrow -i commutator). It follows that the nonvanishing commutators are

$$[z_+, \bar{z}_+] = 1, \quad [z_-, \bar{z}_-] = 1. \quad (130)$$

Note that, the excitation potential is a superposition of two one-dimensional harmonic oscillators. This is very important and will have interesting consequences on the electromagnetic excitations of the quantum Hall effect in four-dimensional phase space.

7.2 Electromagnetic excitations of quantum Hall droplets

The excitation potential (127) determines the spacial shape of the density function

$$\rho_0(\bar{z}_{\pm}, z_{\pm}) = \Theta(M - (\omega_+ z_+ \bar{z}_+ + \omega_- z_- \bar{z}_-)) \quad (131)$$

where M is the total number of particles in the LLL. Following the same strategy as above, one can evaluate the action governing the edge excitations in the presence of an electromagnetic background. The kinetic term in (108) gives

$$i \int dt \text{Tr} \left(\rho_0 U^\dagger \partial_t U \right) \simeq \frac{1}{2} \int \{ \Phi, \rho_0 \} \partial_t \Phi \quad (132)$$

where the symbol $\{, \}$ is the Poisson bracket. It is calculated as

$$\{ \Phi, \rho_0 \} = (\omega_+ \mathcal{L}_+ \Phi + \omega_- \mathcal{L}_- \Phi) \frac{\partial \rho_0}{\partial r^2} \quad (133)$$

where $r^2 = \omega_+ z_+ \bar{z}_+ + \omega_- z_- \bar{z}_-$ and the first order differential operators are defined by

$$\mathcal{L}_\alpha = i \left(z_\alpha \frac{\partial}{\partial z_\alpha} - \bar{z}_\alpha \frac{\partial}{\partial \bar{z}_\alpha} \right), \quad \alpha = +, -. \quad (134)$$

The derivative of the density function, in (133), gives a δ function with support on the boundary $\partial\mathcal{D}$ of the droplet \mathcal{D} defined by $r^2 = M$. Then, we have

$$i \int dt \text{Tr} \left(\rho_0 U^\dagger \partial_t U \right) \approx -\frac{1}{2} \int_{\partial\mathcal{D} \times \mathbf{R}^+} dt (\omega_+ \mathcal{L}_+ \Phi + \omega_- \mathcal{L}_- \Phi) \partial_t \Phi. \quad (135)$$

The evaluation of the potential term in (108) gives

$$\int dt \text{Tr} \left(\rho_0 U^\dagger V U \right) \approx \frac{1}{2} \int_{\partial\mathcal{D} \times \mathbf{R}^+} dt (\omega_+ \mathcal{L}_+ \Phi + \omega_- \mathcal{L}_- \Phi)^2 \quad (136)$$

Combining (132) and (136), we get

$$S \approx -\frac{1}{2} \int_{\partial\mathcal{D} \times \mathbf{R}^+} dt [\omega_+ (\mathcal{L}_+ \Phi) + \omega_- (\mathcal{L}_- \Phi)] [(\partial_t \Phi) + \omega_+ (\mathcal{L}_+ \Phi) + \omega_- (\mathcal{L}_- \Phi)]. \quad (137)$$

This action involves only the time derivative of Φ and tangential derivatives $(\mathcal{L}_\alpha \Phi)$. It is a generalization of a chiral abelian Wess–Zumino–Witten (WZW) theory. For $\theta = 0$ and $\bar{\theta} = 0$, we get

$$S \approx -\frac{1}{2} \int_{\partial\mathcal{D} \times \mathbf{R}^+} dt \left((\partial_t \Phi)(\mathcal{L} \Phi) + \omega (\mathcal{L} \Phi)^2 \right). \quad (138)$$

where $\mathcal{L} = \mathcal{L}_+ + \mathcal{L}_-$. We recover then the WZW usual action for the edge states associated with un-gauged Hall droplets in four-dimensional phase space, see (52). It is interesting to stress that the action (138) is similar to one derived in [2] for the quantum Hall droplets on \mathbf{CP}^2 .

7.3 Edge fields

The action (137) is minimized by the field Φ satisfying the equation of motion

$$\sum_{\alpha=\pm} (\omega_\alpha \mathcal{L}_\alpha) [\partial_t \Phi + \omega_\alpha \mathcal{L}_\alpha \Phi] = 0. \quad (139)$$

The edge field Φ can be expanded in powers of the phase space variables z_α . Note that, since the excitations are moving on the real 3-sphere $\mathbf{S}^3 \sim SU(2)$, it is more appropriate to use the $SU(2)$ parameterization. This is

$$\sqrt{\omega_+} z_+ = \sqrt{M} \frac{\sqrt{\zeta \bar{\zeta}}}{\sqrt{1 + \zeta \bar{\zeta}}} e^{i\phi_+}, \quad \sqrt{\omega_-} z_- = \sqrt{M} \frac{1}{\sqrt{1 + \bar{\zeta} \zeta}} e^{i\phi_-} \quad (140)$$

where ζ and $\bar{\zeta}$ are the local complex coordinates for $SU(2)$. The operators \mathcal{L}_\pm reduce to partial derivatives ∂_{ϕ_\pm} with respect to ϕ_\pm . Thus, the field Φ can be expressed as

$$\Phi = \sum_{n_+, n_-} c_{n_+, n_-}(t) e^{i\phi_+ n_+} e^{i\phi_- n_-} \quad (141)$$

where the coefficients c_{n_+, n_-} are (ϕ_+, ϕ_-) -independents for $(n_+, n_-) \neq (0, 0)$. It follows that the general solution of the equation of motion (139) takes the form

$$\Phi = (\phi_+ - \omega_+ t) + (\phi_- - \omega_- t) + \sum_{n_+, n_-} c_{n_+, n_-}(0) e^{i(\phi_+ - \omega_+ t) n_+} e^{i(\phi_- - \omega_- t) n_-}. \quad (142)$$

It is clear from the last equation that the noncommutativity arising from the symplectic modification changes the propagation velocities of the edge field along the angular directions. It is also important to stress that the velocities ω_+ and ω_- are different (respectively equal) for $\theta \neq \bar{\theta}$ (respectively $\theta = \bar{\theta}$). Finally note that, for $\theta = \bar{\theta} = 0$, i.e. $\omega_{\pm} = 1$, the field (142) coincides with the edge excitations of the \mathbf{CP}^2 quantum Hall droplet derived in [2].

8 Concluding remarks

We introduced the internal phase space of A_n spin systems starting from the analytical representation of A_n spin states. Indeed, the finite dimensional vector spaces, on which the spin matrices act, are provided with an over complete basis labeled by coordinates on the complex projective space \mathbf{CP}^n . As far as the quantum Hall effect on \mathbf{CP}^k is concerned, we investigated the classical phase space structure of A_n particles in the lowest Landau levels. The total phase space incorporates the internal as well as the kinematical degrees of freedom of the Hall system.

For a large collection of A_n particles, forming a dense Hall droplet, we evaluated the action governing the edge excitations. We showed that no trivial dynamics arose and the action is formally similar to that obtained for the spinless case. This agrees with the analysis presented in [13] and is mainly due to the fact that we ignored a possible coupling between the internal and kinematical phase space variables. Indeed, we described A_n spin degrees in terms of an internal compact phase space whose canonical structure is decoupled from the kinematical phase space. In this description, it is clear that any nonexpected spin dynamics can arise out of the Hamiltonian. In this respect, we introduced a nonzero phase structure coupling two spaces. This induces a noncommutative structure in the phase space and can be interpreted as the electromagnetic excitation of the Hall system.

Using a coordinates change based on the Moser's lemma, we obtained an elegant procedure to eliminate the fluctuations of the symplectic two-form. In this way, the effects of the symplectic modifications become incorporated in the Hamiltonian of the system. We derived the bulk and edge actions describing the electromagnetic excitations of dense configuration of A_n particles. It is important to stress that our result agrees with that obtained in [11] for spinless particles. The result presented here deals essentially with symplectic fluctuations.

As illustration, we gave a detailed study for particles, obeying A_1 statistics, living in two-sphere \mathbf{S}^2 . For a specific form of the symplectic fluctuation, we obtained an explicit expression of the edge field excitation. It is clearly shown that the electromagnetic excitations induces an asymmetry between the two angular directions of the droplet. Indeed, the edge field propagates with different velocities along these directions. Finally, we established an interesting relation between the dressing transformation arising from the Moser's lemma and the Seiberg–Witten map.

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